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COMPUTATIONAL METHODS FOR PDES IN FLOW CONTROL, SUPERCONDUCTIVITY, FLUID FLOWS AND OTHER APPLICATIONS

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We give a report on our activities over the period December 1, 1992 –Febuary 28, 1994 supported by AFOSR Grant Number AFOSR F49620-93-1-0061. We give an overview of the research carried out under grant sponsorship and then give details concerning four of the problems we have worked on and for which we have obtained significant results. These are:

- least-squares finite element methods for incompressible, viscous flows;
- analysis of a shape control problem for the Navier-Stokes equations;
- finite dimensional approximation of a class of nonlinear optimal control problems; and
- feedback control of Karman vortex shedding.

We then give lists of papers prepared and personnel supported under grant sponsorship.

RESEARCH RESULTS

During the period of sponsorship by AFOSR Grant Number AFOSR F49620-93-1-0061 we have undertaken and completed a variety of research projects. Here, we briefly describe our research efforts. Below, we will give details concerning four of these projects.

We have undertaken a systematic analytical and computational study of *least-squares finite element methods for incompressible viscous flows*. Details of this research are given below and in papers [1]-[5] listed in the bibliobraphy. We have developed and analyzed the first optimally accurate least-squares finite element method for the Navier-Stokes equations with velocity boundary conditions. Our analyses include the nonlinear case ([2] and [5]) as well as the linear Stokes case ([1], [3], and [4]). In this respect, we have performed perhaps the first rigorous analysis of a least squares finite element method for a nonlinear problem. We have also performed extensive computations which serve to demonstrate the implementation of the techniques, as well as to provide illustrations of the theoretical results.

A second major undertaking is the *optimal control of incompressible viscous flows*. Details are given in papers [6], [8], [10], [14], [15], and [17] listed in the bibliography. We have considered both boundary and shape controls. We have developed and implemented algorithms and analyzed a multidisciplinary optimization problem involving coupled solid/fluid heat transfer [17]. We have an on-going project in the development and implementation of algorithms for the shape control of flows [6]. We have extended our results for the Navier-Stokes case to a model for incompressible viscous flows due to Ladyzhenskaya which features a nonlinear constitutive law [8]. One highlight of our research is that we have given the first rigorous analyses for a shape control problem for the nonlinear Navier-Stokes problem; we provide some details below and in [14] and [15].

As an extension of our work on optimal control of incompressible flows, we have been able to develop and *abstract theory for the finite dimensional approximation of a class of nonlinear optimal control problems*. Details are given below and in the paper [13] listed in the bibliography. This theory gives a list of hypotheses from which one may prove that optimal solutions exists, that Lagrange multipliers may be used to enforce the constraints, and also error estimates for approximations to optimal solutions. We then have shown how problems from disparate areas, i.e., fluids, nonlinear elasticity, and superconductivity, may be analyzed using our abstract theory.

Another area of research has been in the *feedback control of unsteady, viscous, incompressible flows*. Details are given below and in papers [16] and [18] listed in the bibliography. Our main result is to show how a simple feedback law can be effectively used to reduce the size of the oscillations in the lift coefficient in flow about a cylinder This problem is of interest in its own right, e.g., in flows about rotor shafts in helicopters, and is also a prototype for other, more complex problems.

We have also engaged in a study of *models and algorithms for macroscopic supercon-*

ductivity. Details are given in papers [7] and [11] listed in the bibliography. One major accomplishment in this regard is the development, analyses, and computation of a model for superconducting thin films having variable thickness[7]. We have also computed using this model. Two important features of the model is that it is much simpler, and thus cheaper to compute with, that a full-fledged 3D Ginzburg-Landau model and yet it can account for vortex pinning phenomena due to narow regions in the film.

Least-squares finite element methods for incompressible, viscous flows

Least-squares finite element methods for the numerical approximation of solutions of partial differential equations have recently being gaining much attention in the engineering and mathematical communities. In particular, for incompressible viscous flow simulations, such methods applied to first-order formulations are gaining ever increasing interest and acceptance. Some of the reasons that these methods are attractive in the Navier-Stokes setting are:

- the choice of approximating spaces is not subject to the LBB condition that arises in Galerkin mixed finite element approximations;
- a single approximating space can be used for all variables;
- solution methods can be devised that require no matrix assemblies, even at the element level;
- used in conjunction with an appropriate linearization method, e.g., a Newton, results in symmetric, positive definite linear systems, at least in the neighborhood of a solution;
- used in conjunction with properly implemented continuation (with respect to the Reynolds number) techniques, a solution method can be devised that will only encounter symmetric, positive definite linear systems;
- standard and robust iterative methods for symmetric, positive definite linear systems can be used;
- no artificial boundary conditions for the vorticity need be introduced at boundaries at which the velocity is specified; and
- accurate vorticity approximations are obtained.

The goals of our research have been to develop, implement, and analyze least-squares finite element methods for the Stokes and Navier-Stokes equations. In particular, we have developed practical methods that can rigorously be shown to be optimally accurate. As a result, least-squares finite element methods are now demonstrably superior to other methods for incompressible flow simulations in that they require less storage, can be solved for more efficiently, and for the same cost of simulation, yield more accurate results. Here, we will describe some of our results in conjunction with the Stokes and Navier-Stokes equations. Details may be found in [1]-[5].

The best least-squares finite element method for incompressible flow calculations is based on the velocity-vorticity-total pressure formulation. For the generalized linear 2D Stokes equations (we will discuss the 3D and nonlinear cases below), we have the system of first-order differential equations

$$\left.\begin{aligned} \mathbf{curl}\,\omega + \mathbf{grad}\,p &= \mathbf{f}_1 \\ \mathrm{curl}\,\mathbf{u} - \omega &= f_2 \\ \mathrm{div}\,\mathbf{u} &= f_3 \end{aligned}\right\} \quad \text{in } \Omega$$

for the vorticity ω, the velocity $\mathbf{u}$, and the total pressure p. (The pressure itself is easily determined from the total pressure and the velocity.) In the fluids setting, the forcing functions f_2 and f_3 vanish. We consider (and contrast) two types of boundary conditions:

$$\mathbf{u} = \mathbf{U} \qquad \text{on } \Gamma \tag{BC1}$$

or

$$\mathbf{u}\cdot\mathbf{n} = U_n \qquad \text{and} \qquad p = P \qquad \text{on } \Gamma\,. \tag{BC2}$$

We assume that the data satisfy all necessary compatibility conditions, e.g., for the boundary condition (BC1),

$$\int_\Omega f_3\, d\Omega = \int_\Gamma \mathbf{U}\cdot\mathbf{n}\, d\Gamma\,.$$

For simplicity, we will consider homogeneous boundary conditions, i.e., $\mathbf{U} = 0$ or $U_n = 0$ and $P = 0$.

In order to facilitate our discussion, we introduce some *Sobolev space* notation. First, for a positive integer m, we have the space

$$H^m(\Omega) = \{\text{ set of functions such that all partial derivatives of order} \le m \text{ are square integrable }\}$$

and the associated norm for a function g belonging to $H^m(\Omega)$:

$$\|g\|_m^2 = \sum_{m_1+m_2\le m} \int_\Omega \left(\frac{\partial^{(m_1+m_2)} g}{\partial x^{m_1}\partial y^{m_2}}\right)^2 d\Omega\,.$$

For example

$$\|g\|_0^2 = \int_\Omega g^2\, d\Omega\,,$$

$$\|g\|_1^2 = \int_\Omega \left[\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + g^2\right] d\Omega = \|\partial g/\partial x\|_0^2 + \|\partial g/\partial y\|_0^2 + \|g\|_0^2\,, \quad \text{and}$$

$$\|g\|_2^2 = \|\partial^2 g/\partial x^2\|_0^2 + \|\partial^2 g/\partial x\partial y\|_0^2 + \|\partial^2 g/\partial y^2\|_0^2 + \|g\|_1^2\,.$$

Our analyses of the Stokes problem uses the *Agmon-Douglis-Nirenberg* (ADN) theory for elliptic systems of partial differential equations. Of particular interest to use are the following observations about the ADN theory.

- The ADN theory yields a priori estimates for solutions of elliptic boundary value problems.
- The norms appearing in the estimates are chosen so that the differential operator and boundary condition operator satisfy a certain precise condition known as the *complementing condition.*
- If the differential operator is elliptic, and the complementing condition is satisfied, we call the system of partial differential equations and boundary conditions an *ADN system.*
- For the same system of differential equations, different boundary conditions may result in the usage of different norms within the ADN theory, i.e., a system of partial differential equations and boundary conditions may be an ADN system with respect to different norms than the same partial differential equations with different boundary conditions.
- The correct ADN norms are related to the principal part of the differential operator, e.g., the principal part of the operator determines the well-posedeness of the problem.

For the *pressure-normal velocity* boundary condition (BC2), the principal part of the Stokes operator is given by

$$\operatorname{curl}\omega + \operatorname{grad} p, \qquad \operatorname{curl}\mathbf{u}, \qquad \text{and} \qquad \operatorname{div}\mathbf{u}.$$

Only first derivatives appear in the principal part. As a result, one has that all variables have the same differentiability properties. The principal part operator along with the boundary conditions uncouple into two well-posed problems:

$$\begin{aligned} \operatorname{curl}\omega + \operatorname{grad} p = \mathbf{f}_1 &\quad \text{in } \Omega \\ p = P &\quad \text{on } \Gamma \end{aligned}$$

and

$$\begin{aligned} \operatorname{curl}\mathbf{u} = f_2 \quad \text{and} \quad \operatorname{div}\mathbf{u} = f_3 &\quad \text{in } \Omega \\ \mathbf{u}\cdot\mathbf{n} = U_n &\quad \text{on } \Gamma\,. \end{aligned}$$

The ADN a priori estimate relevant to least-squares methods for the pressure-normal velocity boundary condition is given by

$$\|\omega\|_1 + \|p\|_1 + \|\mathbf{u}\|_1 \le C\left(\|\mathbf{f}_1\|_0 + \|f_2\|_0 + \|f_3\|_0\right)\,.$$

Note that all norms on the solution are the same, and that also all the data is measured in just one norm.

If, for the *velocity* boundary condition (BC1), one chooses the same principal part as that chosen above for (BC2), we see that the principal part operator along with the boundary condition uncouple into the two problems

$$\mathbf{curl}\,\omega + \mathbf{grad}\,p = \mathbf{f}_1 \quad \text{in } \Omega$$

and

$$\begin{aligned} \mathrm{curl}\,\mathbf{u} = f_2 \quad \text{and} \quad \mathrm{div}\,\mathbf{u} &= f_3 \quad \text{in } \Omega \\ \mathbf{u} &= \mathbf{U} \quad \text{on } \Gamma\,. \end{aligned}$$

These problems are not well-posed, i.e., the first is underdetermined (not enough boundary conditions), the second is overdetermined (too many boundary conditions). The ADN theory tells us that the principal part of the Stokes operator with velocity boundary conditions is given by

$$\mathbf{curl}\,\omega + \mathbf{grad}\,p\,, \qquad -\omega + \mathrm{curl}\,\mathbf{u}\,, \qquad \text{and} \qquad \mathrm{div}\,\mathbf{u}$$

so that the principal part is the whole Stokes operator. The second operator in the principal part implies that u and ω cannot have the same differentiability properties. The ADN a priori estimate relevant to least-squares methods for the velocity boundary condition is given by

$$\|\omega\|_1 + \|p\|_1 + \|\mathbf{u}\|_2 \le C\,(\|\mathbf{f}_1\|_0 + \|f_2\|_1 + \|f_3\|_1)\,.$$

Note that different components of the solution are measured in different norms and that the different components of the data are also measured in different norms.

Note the consistency achieved by the ADN theory. If u has two square integrable dervatives, then ω and p have one square integrable derivative. Then the combination $\mathbf{curl}\,\omega + \mathbf{grad}\,p$, i.e., $\mathbf{f}_1$, should be merely square integrable, and the combinations $\mathrm{curl}\,\mathbf{u} - \omega$ and $\mathrm{div}\,\mathbf{u}$, i.e., f_2 and f_3, respectively, should have one square integrable derivative. These are exactly the norms appearing in the a priori estimate.

If one uses the *same norm for all unknowns* (and also the same norm for all the data), then, in the velocity boundary condition case, the Stokes system is *not an ADN system*, i.e., the system is not well-posed with respect to those norms.

A *least-squares functional* can be set up by summing up the squares of the residuals of the equations:

$$\mathcal{J}(\mathbf{u},p,\omega) = \|\mathbf{curl}\,\omega + \mathbf{grad}\,p - \mathbf{f}_1\|^2 + \|\mathrm{curl}\,\mathbf{u} - \omega - f_2\|^2 + \|\mathrm{div}\,\mathbf{u} - f_3\|^2\,.$$

Naturally, one asks the question: *what norms should be used to measure the size of the residuals?* An answer:

- if one uses the norms indicated by the ADN theory,
- and if one also uses a conforming finite element method,
- then, from a *practical point of view*, optimally accurate solutions are obtained for all variables

- and, from a *mathematical point of view*, the analysis of errors, e.g., the derivation of rigorous error estimates, is completely straightforward.

For the normal velocity-pressure boundary condition case (BC2), the ADN theory suggests the use of the least-squares functional

$$\begin{aligned}\mathcal{J}(\mathbf{u},p,\omega) &= \|\mathbf{curl}\,\omega + \mathbf{grad}\,p - \mathbf{f}_1\|_0^2 + \|\mathrm{curl}\,\mathbf{u} - \omega - f_2\|_0^2 + \|\mathrm{div}\,\mathbf{u} - f_3\|_0^2 \\ &= \int_\Omega |\mathbf{curl}\,\omega + \mathbf{grad}\,p - \mathbf{f}_1|^2\, d\Omega + \int_\Omega (\mathrm{curl}\,\mathbf{u} - \omega - f_2)^2\, d\Omega + \int_\Omega (\mathrm{div}\,\mathbf{u} - f_3)^2\, d\Omega .\end{aligned}$$

Note that this functional involves at most products of *first* derivatives. The least-squares principle is then given by:

seek $(\mathbf{u},p,\omega)$ *such that* $\mathcal{J}$ *is minimized over an appropriate class of functions* $\mathcal{V}$.

The function class $\mathcal{V}$ consists of $H^1(\Omega)$ velocity, pressure, and vorticity fields, constrained by boundary conditions, etc. The Euler-Lagrange equation for the least squares principle are given by:

seek $(\mathbf{u},p,\omega) \in \mathcal{V}$ *such that* $\mathcal{B}\big((\mathbf{u},p,\omega),\,(\mathbf{v},q,\xi)\big) = \mathcal{F}(\mathbf{v},q,\xi)$ for all $(\mathbf{v},q,\xi) \in \mathcal{V}$,

where

$$\begin{aligned}\mathcal{B}\big((\mathbf{u},p,\omega),\,(\mathbf{v},q,\xi)\big) &= \int_\Omega (\mathbf{curl}\,\omega + \mathbf{grad}\,p)\cdot(\mathbf{curl}\,\xi + \mathbf{grad}\,q)\, d\Omega \\ &\quad + \int_\Omega (\mathrm{curl}\,\mathbf{u} - \omega)(\mathrm{curl}\,\mathbf{v} - \xi)\, d\Omega + \int_\Omega (\mathrm{div}\,\mathbf{u})(\mathrm{div}\,\mathbf{v})\, d\Omega\end{aligned}$$

and

$$\mathcal{F}(\mathbf{v},q,\xi) = \int_\Omega \mathbf{f}_1\cdot(\mathbf{curl}\,\xi + \mathbf{grad}\,q)\, d\Omega + \int_\Omega f_2(\mathrm{curl}\,\mathbf{v} - \xi)\, d\Omega + \int_\Omega f_3 \mathrm{div}\,\mathbf{v}\, d\Omega .$$

Conforming finite element approximations are defined in the usual manner. One chooses a conforming finite element approximating space $\mathcal{V}^h$, i.e., the finite element functions for all variables have one square integrable derivative. Then, one poses the problem:

$$\textit{seek } (\mathbf{u}^h,p^h,\omega^h) \in \mathcal{V}^h \textit{ such that}$$
$$\mathcal{B}\big((\mathbf{u}^h,p^h,\omega^h),\,(\mathbf{v}^h,q^h,\xi^h)\big) = \mathcal{F}(\mathbf{v}^h,q,\xi^h) \textit{ for all } (\mathbf{v}^h,q^h,\xi^h) \in \mathcal{V}^h .$$

This problems is equivalent to a linear algebraic system having a symmetric, positive definite coefficient matrix. Standard finite element methodology, i.e., based on the Lax-Milgram theorem, can be used to derive optimal error estimates. For example, if piecewise linear polynomials are used for all variables (and the exact solution is sufficiently smooth), one finds that

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}^h\|_1 + \|p-p^h\|_1 + \|\omega-\omega^h\|_1 &= O(h)\\ \|\mathbf{u}-\mathbf{u}^h\|_0 + \|p-p^h\|_0 + \|\omega-\omega^h\|_0 &= O(h^2);\end{aligned}$$

if piecewise quadratic polynomials are used, one finds that

$$\|\mathbf{u}-\mathbf{u}^h\|_1 + \|p-p^h\|_1 + \|\omega-\omega^h\|_1 = O(h^2)$$
$$\|\mathbf{u}-\mathbf{u}^h\|_0 + \|p-p^h\|_0 + \|\omega-\omega^h\|_0 = O(h^3)\,.$$

Note that all variables are approximated by the same finite element functions, all variables are optimally approximated, and conforming finite element approximations for all variables are required to be merely continuous across element edges.

For the velocity boundary condition case (BC1), the ADN theory suggests the use of the least-squares functional

$$\begin{aligned}
\mathcal{K}(\mathbf{u},p,\omega) &= \|\mathbf{curl}\,\omega + \mathbf{grad}\,p - \mathbf{f}_1\|_0^2 + \|\mathrm{curl}\,\mathbf{u} - \omega - f_2\|_1^2 + \|\mathrm{div}\,\mathbf{u} - f_3\|_1^2 \\
&= \int_\Omega |\mathbf{curl}\,\omega + \mathbf{grad}\,p - \mathbf{f}_1|^2\, d\Omega + \int_\Omega |\mathrm{grad}\ (\mathrm{curl}\,\mathbf{u} - \omega - f_2)|^2 \\
&\quad + \int_\Omega (\mathrm{curl}\,\mathbf{u} - \omega - f_2)^2\, d\Omega + \int_\Omega \Big(|\mathrm{grad}\ (\mathrm{div}\,\mathbf{u} - f_3)|^2 + (\mathrm{div}\,\mathbf{u} - f_3)^2 \Big)\, d\Omega
\end{aligned}$$

Note that this functional involves products of *second* derivatives of the velocity. The least-squares principle is then given by:

seek $(\mathbf{u},p,\omega)$ *such that* $\mathcal{K}$ *is minimized over an appropriate class of functions* $\mathcal{W}$.

The function class $\mathcal{W}$ now consists of $H^1(\Omega)$ pressure and vorticity fields and $H^2(\Omega)$ velocity fields, constrained by boundary conditions. The Euler-Lagrange equation for this least squares principle again has the form

seek $(\mathbf{u},p,\omega) \in \mathcal{W}$ *such that* $\mathcal{B}\big((\mathbf{u},p,\omega)\,,\,(\mathbf{v},q,\xi)\big) = \mathcal{F}(\mathbf{v},q,\xi)$ *for all* $(\mathbf{v},q,\xi) \in \mathcal{W}$,

where now $\mathcal{B}$ involves products of second derivatives of $\mathbf{u}$ and $\mathbf{v}$.

Conforming finite element approximations are again defined in the usual manner. One chooses a conforming finite element approximating space $\mathcal{W}^h$, i.e., one chooses the finite element functions for approximating the pressure and vorticity so that their derivatives are square integrable and finite element functions for approximating the velocity so that their *second* derivatives are square integrable. Then, one poses the approximate problem:

seek $(\mathbf{u}^h,p^h,\omega^h) \in \mathcal{W}^h$ *such that*
$\mathcal{B}\big((\mathbf{u}^h,p^h,\omega^h)\,,\,(\mathbf{v}^h,q^h,\xi^h)\big) = \mathcal{F}(\mathbf{v}^h,q,\xi^h)$ *for all* $(\mathbf{v}^h,q^h,\xi^h) \in \mathcal{W}^h$.

This problems is again equivalent to a linear algebraic system having a symmetric, positive definite coefficient matrix. Standard finite element methodology, i.e., based on the Lax-Milgram theorem, can be used to derive optimal error estimates whenever conforming finite element spaces are used. However, *the method is not practical.* The requirement that finite element velocity approximations possess two square integrable derivatives forces one

to use finite element functions that are *continuously differentiable* across element edges. (Incidentally, if one is willing to use continuously differentiable velocity approximations, one might as well do least-squares on the primitive variable formulation!)

Thus we are faced with the following dilemma. If, for velocity boundary conditions, we use a least-squares functional based on the ADN norms we are led to a computational method requiring continuously differentiable velocity approximations, i.e., *we get an impractical method.* If instead we use the more practical functional $\mathcal{J}$ that works easily and optimally for the normal velocity/pressure boundary conditions, *we get non-optimal approximations.* Naturally, one may ask the following question:

is there a way to use the simpler and more practical norms of the functional $\mathcal{J}$ and still get optimally accurate approximations?

An affirmative answer is found by introducing mesh-dependent weights in the least-squares functional. The residual norms we would like to use are given by

$$\|\mathbf{curl}\,\omega + \mathbf{grad}\,p - \mathbf{f}_1\|_0\,, \qquad \|\mathrm{curl}\,\mathbf{u} - \omega - f_2\|_0\,, \qquad \text{and} \qquad \|\mathrm{div}\,\mathbf{u} - f_3\|_0\,.$$

The residual norms the ADN theory would like us to use

$$\|\mathbf{curl}\,\omega + \mathbf{grad}\,p - \mathbf{f}_1\|_0\,, \qquad \|\mathrm{curl}\,\mathbf{u} - \omega - f_2\|_1\,, \qquad \text{and} \qquad \|\mathrm{div}\,\mathbf{u} - f_3\|_1\,.$$

A standard *inverse inequality* for finite element functions is given by

$$\|q^h\|_1 \le Ch^{-1}\|q^h\|_0\,,$$

where h is an appropriate measure of the grid size. This suggests that, for finite element functions, one can "simulate" the norm $\|q^h\|_1$ by $h^{-1}\|q^h\|_0$. Thus, the weighted least squares functional is given by

$$\begin{aligned}
\mathcal{J}_h(\mathbf{u},p,\omega) &= \|\mathbf{curl}\,\omega + \mathbf{grad}\,p - \mathbf{f}_1\|_0^2 + h^{-2}\|\mathrm{curl}\,\mathbf{u} - \omega - f_2\|_0^2 + h^{-2}\|\mathrm{div}\,\mathbf{u} - f_3\|_0^2 \\
&= \int_\Omega |\mathbf{curl}\,\omega + \mathbf{grad}\,p - \mathbf{f}_1|^2\, d\Omega \\
&\qquad + \frac{1}{h^2}\int_\Omega (\mathrm{curl}\,\mathbf{u} - \omega - f_2)^2\, d\Omega + \frac{1}{h^2}\int_\Omega (\mathrm{div}\,\mathbf{u} - f_3)^2\, d\Omega\,.
\end{aligned}$$

This functional uses the norms that lead to a simple, easy to implement algorithm for which one can use *merely continuous* finite element functions for all variables and for which one still obtains a *symmetric, positive definite* discrete linear system. This functional also leads to *optimally accurate* approximations. Note that optimal accuracy is achieved with respect to norms related to the ADN theory for velocity boundary conditions.

The finite element analyses indicate that one may use polynomials of one degree lower for the pressure and vorticity than one uses for the velocity. If we use continuous piecewise quadratic polynomials for the velocity approximations and piecewise linear polynomials for the pressure and vorticity approximations, we get the estimate

$$\|\mathbf{u} - \mathbf{u}^h\|_1 + \|p - p^h\|_0 + \|\omega - \omega^h\|_0 = O(h^2)\,.$$

This estimate is optimal with respect to the finite element functions used. The theory also says that one may use the same degree polynomials for all variables. For example, if one uses continuous piecewise quadratic polynomials for all variables, one again obtains the above error estimate. In this case, the above estimate is not optimal for the pressure and vorticity.

In *three dimensions*, the velocity-vorticity-pressure formulation of the Stokes equations with velocity boundary conditions is given by

$$\left.\begin{aligned} \mathrm{curl}\,\omega + \mathrm{grad}\,p &= \mathbf{f}_1 \\ \mathrm{curl}\,\mathbf{u} - \omega &= \mathbf{f}_2 \\ \mathrm{div}\,\mathbf{u} &= f_3 \end{aligned}\right\} \quad \text{in } \Omega$$

and

$$\mathbf{u} = \mathbf{U} \qquad \text{on } \Gamma\,,$$

where now the vorticity ω is a vector valued function. We immediately have a problem: we have seven equations in seven unknowns so that the system cannot be elliptic. We avoid this problem by adding an extra variable ϕ and a seemingly redundant equation to get the elliptic system of 8 equations in 8 unknowns:

$$\left.\begin{aligned} \mathrm{curl}\,\omega + \mathrm{grad}\,p &= \mathbf{f}_1 \\ \mathrm{div}\,\omega &= -\mathrm{div}\,\mathbf{f}_2 \\ \mathrm{curl}\,\mathbf{u} + \mathrm{grad}\,\phi - \omega &= \mathbf{f}_2 \\ \mathrm{div}\,\mathbf{u} &= f_3 \end{aligned}\right\} \quad \text{in } \Omega\,.$$

One can easily show that $\phi = 0$. Analyses yield the same results as in the 2D case. Algorithmically, one can completely ignore ϕ, but one cannot ignore the redundant equation. Thus, we can discretize the problem

$$\left.\begin{aligned} \mathrm{curl}\,\omega + \mathrm{grad}\,p &= \mathbf{f}_1 \\ \mathrm{div}\,\omega &= -\mathrm{div}\,\mathbf{f}_2 \\ \mathrm{curl}\,\mathbf{u} - \omega &= \mathbf{f}_2 \\ \mathrm{div}\,\mathbf{u} &= f_3 \end{aligned}\right\} \quad \text{in } \Omega$$

$$\mathbf{u} = \mathbf{U} \qquad \text{on } \Gamma$$

using the weighted least squares functional

$$\begin{aligned} \mathcal{J}_h(\mathbf{u}, p, \omega) &= \|\mathrm{curl}\,\omega + \mathrm{grad}\,p - \mathbf{f}_1\|_0^2 + \|\mathrm{div}\,\omega + \mathrm{div}\,\mathbf{f}_2\|_0^2 \\ &\qquad + h^{-2}\|\mathrm{curl}\,\mathbf{u} - \omega - \mathbf{f}_2\|_0^2 + h^{-2}\|\mathrm{div}\,\mathbf{u} - f_3\|_0^2 \\ &= \int_\Omega |\mathrm{curl}\,\omega + \mathrm{grad}\,p - \mathbf{f}_1|^2 \, d\Omega + \int_\Omega (\mathrm{div}\,\omega + \mathrm{div}\,\mathbf{f}_2)^2 \, d\Omega \\ &\qquad + \frac{1}{h^2}\int_\Omega (\mathrm{curl}\,\mathbf{u} - \omega - \mathbf{f}_2)^2 \, d\Omega + \frac{1}{h^2}\int_\Omega (\mathrm{div}\,\mathbf{u} - f_3)^2 \, d\Omega \end{aligned}$$

The above discussion can be extended to the *nonlinear Navier-Stokes equations*

$$\left.\begin{aligned} \nu \operatorname{curl} \omega + \operatorname{grad} p + \omega \times \mathrm{u} = \mathrm{f}_1 \\ \operatorname{curl} \mathrm{u} - \omega = 0 \\ \operatorname{div} \mathrm{u} = 0 \end{aligned}\right\} \quad \text{in } \Omega$$

and

$$\mathrm{u} = \mathrm{U} \qquad \text{on } \Gamma .$$

The weighted least squares functional in this case is given by

$$\begin{aligned} \mathcal{J}_h(\mathrm{u}, p, \omega) &= \nu^{-2} \|\nu \operatorname{curl} \omega + \operatorname{grad} p + \omega \times \mathrm{u} - \mathrm{f}_1\|_0^2 + \nu^{-2} \|\operatorname{div} \omega\|_0^2 \\ &\qquad + h^{-2} \|\operatorname{curl} \mathrm{u} - \omega\|_0^2 + h^{-2} \|\operatorname{div} \mathrm{u}\|_0^2 \\ &= \frac{1}{\nu^2} \int_\Omega |\nu \operatorname{curl} \omega + \operatorname{grad} p + \omega \times \mathrm{u} - \mathrm{f}_1|^2 \, d\Omega + \frac{1}{\nu^2} \int_\Omega |\operatorname{div} \omega|^2 \, d\Omega \\ &\qquad + \frac{1}{h^2} \int_\Omega (\operatorname{curl} \mathrm{u} - \omega)^2 \, d\Omega + \frac{1}{h^2} \int_\Omega (\operatorname{div} \mathrm{u})^2 \, d\Omega . \end{aligned}$$

All theoretical results for the Stokes equations can also be derived in the context of the nonlinear Navier-Stokes equations. Algorithmically, one must choose a method for linearizing the equation. If one uses Newton's method, then in the neighborhood of a solution, the Hessian matrix is not only symmetric, but is also positive definite.

The analysis of the nonlinear case required and extension of the Brezzi-Rappaz-Raviart theory for the approximation of nonlinear problems. That theory addresses problems of the following type. We let X and Y denote Banach spaces, Λ a compact interval of the real line, $T : Y \to X$ a linear operator, and $G : X \times \Lambda \to Y$ a nonlinear operator. Then, for $\lambda \in \Lambda$, we seek $\psi \in X$ such that

$$\psi + TG(\psi, \lambda) = 0 .$$

Approximations are defined as follows. Let $X^h \subset X$ and let $T^h : Y \to X^h$ be a linear operator. Then, for $\lambda \in \Lambda$, we seek $\psi^h \in X^h$ such that

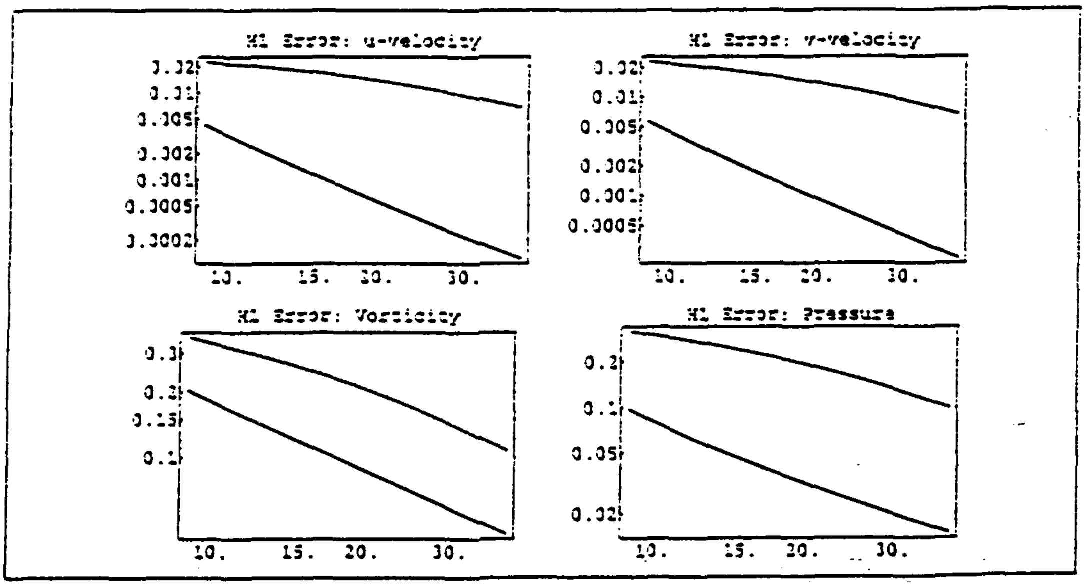
$$\psi^h + T^h G(\psi^h, \lambda) = 0 .$$

Under certain conditions on the G, it turns out that estimates for the error $\|\psi - \psi^h\|_X$ can be determined from estimates for $\|T - T^h\|$. This theory has been widely applied; for example, it can be used to determine error estimates for Galerkin mixed finite element approximations of the primitive variable formulation of the Navier-Stokes equations. It can also be applied directly to conforming least squares finite element approximations of the velocity-vorticity-pressure formulation of the Navier-Stokes equations based on the use of the norms suggested by the ADN theory. In these cases, the operator T is merely the solution operator of the corresponding Stokes problem and an estimate for $\|T - T^h\|$ can be obtained from the analyses of the analogous least-squares finite element approximation

of the Stokes equations. One finds that the errors in approximations of solutions of the nonlinear Navier-Stokes equations behave in the same manner (with respect to the grid size) as do the errors in the analogous approximations of solutions of the linear Stokes equations.

Least-squares finite element methods for the Navier-Stokes equations based on the use of weighted functionals do not fit into the Brezzi-Rappaz-Raviart framework. The problem fits into the form $\psi + TG(\psi, \lambda) = 0$ using the space X suggested by the ADN theory. However, although the dicsrete problem is of the form $\psi^h + T^h G(\psi^h, \lambda) = 0$, the discrete space $X^h \not\subset X$, i.e., we are within the realm of nonconforming finite element methods. We have extended the Brezzi-Rappaz-Raviart theory to the nonconforming case and have shown, for least-squares finite element methods for the Navier-Stokes equations based on wieghted functionals that optimal error estimates are obtained.

We give the results of some computational experiments merely to show the necessity of introducing the weights into the functional. For these figures, we use piecewise linear pressures and vorticities and piecewise quadratic velocities. (Similar results can be obtained for piecewise quadratic velocities and pressures.) In the figures below, there are two graphs in each frame. The one with the steeper slope corresponds to the use of the weighted functional, while the other one corresponds to the use of the unweighted functional. Thus, we see that the use of the unweighted functional results in a loss of accuracy. Morever, one can verify that the slope of the graphs correponding to the use of the weighted functional indicate that those approximations converge at an optimal rate.



Analysis of a shape control problem for the Navier-Stokes equations

Shape controls are of great interest in the control of fluid flows. For example, the problem of the optimal design of wings is a shape optimal control problem. There have

been substantial analyses of linear shape control problems for partial differential equations. However, for the nonlinear Navier-Stokes equations of incompressible flow, no rigorous analyses have been previously carried out. We have given the first such analyses in the context of a two-dimesional flow in a channel-like domain. Our rigorous analyses include showing, in the context of a drag minimization problem, that optimal solutions exist, showing that the Lagrange multplier may be used to enforce constraints, deriving an otpimality system from which optimal states and co-states may be deduced, deriving the shape gradient, and finally, deriving error estimates for finite element approximations of optimal states. Details will be provided in [14] and [15].

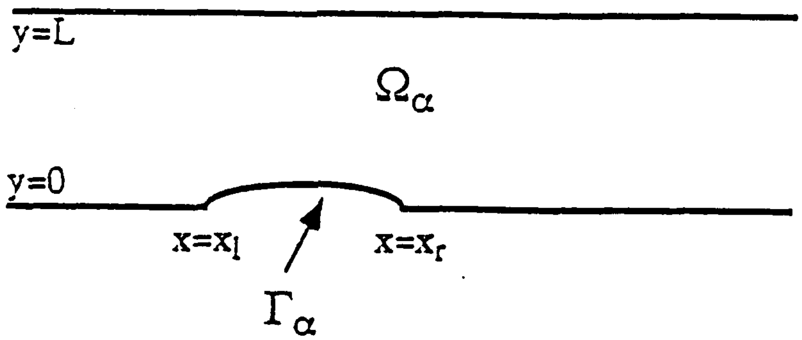
Our prototype problem is that of two-dimensional incompressible flow of a viscous fluid passing through a channel having a bump on the lower wall; see the figure below. (Our results can be extended to a variety of other problems.) We describe the bump through the function $y = \alpha(x)$; note that the extent of the bump $[x_\ell, x_r]$ is fixed and that $\alpha(x_\ell) = \alpha(x_r) = 0$ are imposed as constraints on $\alpha(x)$. We also require that $\alpha(x) \leq L$. Thus, the bump is described by

$$\Gamma_\alpha = \{ (x,y) \in [x_\ell, x_r] \times [0, L] \ : \ y = \alpha(x) \} .$$

We let Γ denote the boundary of the flow domain Ω_α; of course, $\Gamma_\alpha \subset \Gamma$. The *admissible family of curves* Γ_α is defined by

$$\mathcal{U}_{ad} = \{ \alpha \in C^{0,1}(x_\ell, x_r) \ : \ 0 \leq \alpha(x) \leq L \, , \ |\alpha'(x)| \leq \beta \, \forall x \in [x_\ell, x_r] \, , \ \alpha(x_\ell) = \alpha(x_r) = 0 \} \, ,$$

where the constant $\beta > 0$ is chosen so that $\mathcal{U}_{ad} \neq \emptyset$ and $C^{0,1}(x_\ell, x_r)$ denotes the Lipschitz continuous functions defined on $[x_\ell, x_r]$.



The *state equations* are defined on Ω_α for each $\alpha \in \mathcal{U}_{ad}$. They are given by the Navier-Stokes system:

$$-\nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{f} \qquad \text{in } \Omega_\alpha \, ,$$

$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega_\alpha \, ,$$

and

$$\mathbf{u} = \begin{cases} \mathbf{g} & \text{on } \Gamma / \Gamma_\alpha \\ 0 & \text{on } \Gamma_\alpha \, , \end{cases}$$

where f and g ar given fucntions. (Other boundary conditions on Γ/Γ_α can also be treated.)

The *objective functional* we treat is the dissipation functional

$$\mathcal{J}(\alpha) = \mathcal{J}(\Omega_\alpha, \mathbf{u}_\alpha) = \nu \int_{\Omega_\alpha} \nabla \mathbf{u}_\alpha : \nabla \mathbf{u}_\alpha \, dx \,,$$

where $\mathbf{u}(\alpha)$ is a solution of the state equations in the domain Ω_α. (We could also consider other functionals such as those involving the matching of the flow to some given flow.)

The *extremal problem* we consider is then given by:

$$\min_{\alpha \in \mathcal{U}_{ad}} \mathcal{J}(\alpha) \quad \text{subject to} \quad (\mathbf{u}_\alpha, p_\alpha) \quad \text{satisfying the Navier-Stokes system over } \Omega_\alpha \,.$$

Our first main result is to show that this extremal problem has a solution. We then show that the use of the Lagrange multiplier rule to enforce the contraints, i.e., the Navier-Stokes system, is rigorously justified. Subsequently, we derive the necessary conditions that optimal controls and states must satisfy. These are found by requiring that variations in an appropriate Lagrangian vanish. Variations with respect to the Lagrange multipliers yields, of course, the state constraint equations, i.e., the Navier-Stokes system. Variations in the state $(\mathbf{u}, p)$ yield the co-state equations

$$-\nu \Delta \mathbf{q} - \mathbf{u} \cdot \nabla \mathbf{q} + \mathbf{q} \cdot (\nabla \mathbf{q})^T + \nabla r = -2\nu \Delta \mathbf{u} \qquad \text{in } \Omega_\alpha \,,$$

$$\nabla \cdot \mathbf{q} = 0 \qquad \text{in } \Omega_\alpha \,,$$

and

$$\mathbf{q} = 0 \qquad \text{on } \Gamma \,.$$

Note that, as usual, these are the adjoint of the linearized Navier-Stokes system where the linearization is about the state u. Variations in the control function α can be used to determine the shape gradient of the functional $\mathcal{J}$, i.e., roughly speaking, the derivative of $\mathcal{J}$ with respect to α. This is given by

$$grad\mathcal{J} = \nu \nabla \mathbf{u} : \nabla \mathbf{u} - \nu \nabla \mathbf{u} : \nabla \mathbf{q} + p \nabla \cdot \mathbf{q} - (\mathbf{n} \cdot \nabla \mathbf{u}) \cdot \Big(2\nu (\mathbf{n} \cdot \nabla \mathbf{u}) - \nu (\mathbf{n} \cdot \nabla \mathbf{q}) + r \mathbf{n} \Big) \,.$$

Thus, we see that, for a given function α, the shape gradient $grad\mathcal{J}$ depends on the state $(\mathbf{u}, p)$ and the adjoint state $(\mathbf{q}, r)$ determined from the state and co-state systems posed over the domain Ω_α, i.e., for a given α, $grad\mathcal{J}(\alpha) = grad\mathcal{J}(\mathbf{u}_\alpha, p_\alpha, \mathbf{q}_\alpha, r_\alpha)$. (The above presentation is formal; the rigorous derivations we have done are with respect to weak formulations and solutions posed over appropriate function spaces.)

A typical optimization algorithm would use the shape greadient $grad\mathcal{J}$, as well as $\mathcal{J}$ itself, to update a guess for the shape function α. As we have seen, we may compute the shape gradient by for any iterate α by solving the state equations, i.e, the Navier-Stokes system, and the co-state equations with respect to the domain Ω_α. Then, these solutions

are used to evaluate the functional and the shape gradient of the functional $\mathcal{J}$ using the above formulas. Thus, the above equations completely determine the information needed by any optimization algorithm requiring gradient information. Our main contributions in this respect are to derive the sjhape gradient formula and to rigorously justify all the steps performed in that derivation.

Of course, in general one can only solve the state and co-state systems in an approximate manner. To this end we have defined finite element methods for the approximation of the state and co-state equations (for a given value of α), and derived optimal error estimates for the finite element approximations. We have also defined approximations for the stress and adjoint stress vectors (these appear in the definition of the shape gradient) and derived optimal error estimates for these quantities in appropriate norms. Thus, we also have error estimates for approximations of the shape gradient.

Finite dimensional approximation of a class of nonlinear optimal control problems

The need to solve optimization and control problems arises in many settings. Although in some cases these problems may be easily solved, either analytically or computationally, in many other cases substantial difficulties are encountered. For example, candidate optimal states and controls may belong to infinite dimensional function spaces and one may have to minimize a nonlinear functional of the state and control variables subject to nonlinear constraints that take the form of a system of partial differential equations whose solutions are in general not unique. Our goal, which we have reached (see [13] for details), was to construct, analyze, and apply a framework for the approximate solution of many such problems. The setting for our framework is a class of nonlinear control or optimization problems which is general enough to be of use in numerous applications. The major steps in the development and analysis of our framework are as follows:

- define an abstract class of nonlinear control or optimization problems;
- show that, under certain assumptions, optimal solutions exist;
- show that, under certain additional assumptions, Lagrange multipliers exist that may be used to enforce the constraints;
- use the Lagrange multiplier technique to derive an optimality system from which optimal states and controls may be deduced;
- define algorithms for the finite dimensional approximation of optimal states and controls; and
- derive estimates for the error in the approximations to the optimal states and controls.

Two of the key ingredients used to carry out the above plan are the Tikhomorov version of the Lagrange multiplier theory and the Brezzi-Rappaz-Raviart theory for the approxima-

tion of a class of nonlinear problems. In both of these theories, certain properties of compact operators on Banach spaces play a central role. We point out that the nonuniqueness of solutions of the nonlinear constraint equations deems it appropriate to employ Lagrange multiplier principles.

After having developed and analyzed the abstract framework, we applied it to some specific, concrete problems. In each case, we used the abstract framework to analyze the concrete problems by merely showing that the latter fit into the former. The particular applications we considered are:

- control problems in structural mechanics having geometric nonlinearities that are governed by the von Kármán equations;
- control problems in superconductivity that are governed by the Gin r-Landau equations; and
- control problems for incompressible, viscous flows that are governed by the Navier-Stokes equations.

In considering these applications, we will purposely chose different types of controls in order to illustrate how these could be accounted for within the abstract framework. In all three cases, approximation were effected through the use of finite element methods.

We now give a precise definition of the abstract class of nonlinear control or optimization problems that we have studied and for which the above results have been obtained. We introduce the spaces and control set as follows. Let G, X, and Y be reflexive Banach spaces whose norms are denoted by $\|\cdot\|_G$, $\|\cdot\|_X$, and $\|\cdot\|_Y$, respectively. Dual spaces will be denoted by $(\cdot)^*$. The duality pairing between X and X^* is denoted by $\langle\cdot,\cdot\rangle_X$; one similarly defines $\langle\cdot,\cdot\rangle_Y$ and $\langle\cdot,\cdot\rangle_G$. The subscripts are often omitted whenever there is no chance for confusion. Let Θ, the control set, be a closed convex subset of G. Let Z be a subspace of Y with a compact imbedding. Note that the compactness of the imbedding $Z \subset Y$ plays an important role. We assume that the *functional* to be minimized takes the form

$$\mathcal{J}(v,z) = \lambda\,\mathcal{F}(v) + \lambda\,\mathcal{E}(z) \quad \forall\,(v,z) \in X \times \Theta\,,$$

where $\mathcal{F}$ is a functional on X, $\mathcal{E}$ a functional on Θ, and λ is a given parameter which is assumed to belong to a compact interval $\Lambda \subset \mathbf{R}_+$. The *constraint equation* $M(v,z) = 0$ relating the state variable v and the control variable z is defined as follows. Let N be a differentiable mapping from X to Y, K a continuous linear operator from Θ to Y, and T a continuous linear operator from Y to X. For any $\lambda \in \Lambda$, we define the mapping M from $X \times \Theta$ to X by

$$M(v,z) = v + \lambda TN(v) + \lambda TK(z) \quad \forall\,(v,z) \in X \times \Theta\,.$$

With these definitions we now consider the constrained minimization problem:

$$\min_{(v,z)\in X\times\Theta} \mathcal{J}(v,z) \quad \text{subject to} \quad M(v,z) = 0\,.$$

Here, we are seeking a global minimizer with respect to the set $\{(v,z) \in X \times \Theta : M(v,z) = 0\}$. Although, under suitable hypotheses, we have shown that this problem has a solution, in practice, one can only characterize local minima, i.e., points $(u,g) \in X \times \Theta$ such that for some $\epsilon > 0$

$$\mathcal{J}(u,g) \leq \mathcal{J}(v,z) \quad \forall\, (v,z) \in X \times \Theta \text{ such that } \quad M(v,z) = 0 \text{ and } \|u - v\|_X \leq \epsilon .$$

Thus, when algorithms for locating constrained minima of $\mathcal{J}$ are considered, one must be content to find local minima.

After having showed that *optimal solutions exist* and that *one is rigorously justified in using the Lagrange multiplier rule,* we introduce some simplifications in order to render the abstract problem more amenable to approximation. The first is to only consider the control set $\Theta = G$. The second is to only consider Fréchet differentiable functionals $\mathcal{E}(\cdot)$ such that the Fréchet derivative $\mathcal{E}'(g) = E^{-1}g$, where E is an invertible linear operator from G^* to G.

In order to prove our results, we will have to introduce certain hypotheses. The first set of hypotheses are invoked to prove the existence of optimal solutions. It is given by:

(H1) $\inf_{v \in X} \mathcal{F}(v) > -\infty$;

(H2) *there exist constants* α, $\beta > 0$ *such that* $\mathcal{E}(z) \geq \alpha \|z\|^\beta \ \forall\, z \in \Theta$;

(H3) *there exists a* $(v,z) \in X \times \Theta$ *satisfying* $M(v,z) = 0$;

(H4) *if* $u^{(n)} \rightharpoonup u$ *in* X *and* $g^{(n)} \rightharpoonup g$ *in* G *where* $\{(u^{(n)}, g^{(n)})\} \subset X \times \Theta$, *then* $N(u^{(n)}) \rightharpoonup N(u)$ *in* Y *and* $K(g^{(n)}) \rightharpoonup K(g)$ *in* Y;

(H5) $\mathcal{J}(\cdot,\cdot)$ *is weakly lower semicontinuous on* $X \times \Theta$; *and*

(H6) *if* $\{(u^{(n)}, g^{(n)})\} \subset X \times \Theta$ *is such that* $\{\mathcal{F}(u^{(n)})\}$ *is a bounded set in* $\mathbb{R}$ *and* $M(u^{(n)}, g^{(n)}) = 0$, *then* $\{u^{(n)}\}$ *is a bounded set in* X.

The second set of assumptions are used to justify the use of the Lagrange multiplier rule and to derive an optimality system from which optimal states and controls may be determined. The second set is given by:

(H7) *for each* $z \in \Theta$, $v \mapsto \mathcal{J}(v,z)$ *and* $v \mapsto M(v,z)$ *are Fréchet differentiable*;

(H8) $z \mapsto \mathcal{E}(z)$ *is convex, i.e.,*

$$\mathcal{E}(\gamma z_1 + (1-\gamma) z_2) \leq \gamma\, \mathcal{E}(z_1) + (1-\gamma)\, \mathcal{E}(z_2) \quad \forall\, z_1, z_2 \in \Theta,\ \forall\, \gamma \in [0,1];$$

and

(H9) *for* $v \in X$, $N'(v)$ *maps* X *into* Z.

In (H9), N' denotes the Fréchet derivative of N. A simplified optimality system may be obtained if one invokes the additional assumption:

(H10) $\Theta = G$, *and the mapping* $z \mapsto \mathcal{E}(z)$ *is Fréchet differentiable on* G.

Hypotheses (H7)-(H10) allow us to obtain a simplified optimality system for almost all values of the parameter $\lambda \in \Lambda$. In many cases, it is possible to show that the same optimality system holds for all values of λ. The following two additional assumptions which will only be invoked in case $(1/\lambda)$ is an eigenvalue of $-TN'(u)$ each provides a setting in which this last result is valid:

(H11) *if $v^* \in X^*$ satisfies $(I + \lambda [N'(u)]^* T^*) v^* = 0$ and $K^* T^* v^* = 0$, then $v^* = 0$; or*

(H11)′ *the mapping $(v, z) \mapsto v + \lambda TN'(u)v + \lambda TKz$ is onto from $X \times G$ to Y.*

In order to make the optimality system more amenable to approximation and computation, we invoke the following additional assumption:

(H12) *$\mathcal{E}'(g) = E^{-1}g$, where E is an invertible linear operator from G^* to G and g is an optimal control for the constrained minimization problem* (2.4).

Assumptions (H1)-(H6) can be used to establish that optimal solutions exist. With the addition of (H7)-(H9), one can show that the Lagrange multiplier rule may be used to turn the constrained minimization problem into an unconstrained one and to derive an *optimality system* from which optimal states and controls may be determined. By also invoking (H10), one of (H11) or (H11'), and (H12) a simplified optimality system is achieved. In this case, an optimal state $u \in X$, an optimal control $g \in G$, and the corresponding Lagrange multiplier $\xi \in Y^*$ satisfy the *optimality system*

$$u + \lambda TN(u) + \lambda TKg = 0 \quad \text{in } X,$$

$$\xi + \lambda T^*[N'(u)]^*\xi - \lambda T^*\mathcal{F}'(u) = 0 \quad \text{in } Y^*,$$

and

$$g - EK^*\xi = 0 \quad \text{in } G.$$

In many applications we have that $X^* = Y$. Since these spaces are assumed to be reflexive, we also have that $Y^* = X$. In this case, we have that both u and ξ belong to X.

A finite dimensional discretization of the optimality system is defined as follows. First, one chooses families of finite dimensional subspaces $X^h \subset X$, $(Y^*)^h \subset Y^*$, and $G^h \subset G$. These families are parameterized by a parameter h that tends to zero. (For example, this parameter can be chosen to be some measure of the grid size in a subdivision of Ω into finite elements.) Next, we define approximate operators $T^h : Y \to X^h$, $E^h : G^* \to G^h$, and $(T^*)^h : X^* \to (Y^*)^h$. Of course, one views T^h, E^h, and $(T^*)^h$ as approximations to the operators T, E, and T^*, respectively. Note that $(T^*)^h$ is not necessarily the same as $(T^h)^*$. The former is a discretization of an adjoint operator while the later is the adjoint of a discrete operator. Once the approximating subspaces and operators have been chosen, an approximate problem, or the *discrete optimality system*, is defined as follows. We seek $u^h \in X^h$, $g^h \in G^h$, and $\xi^h \in (Y^*)^h$ such that

$$u^h + \lambda T^h N(u^h) + \lambda T^h K g^h = 0 \quad \text{in } X^h,$$

$$\xi^h + \lambda (T^*)^h [N'(u^h)]^* \xi^h - \lambda (T^*)^h \mathcal{F}'(u^h) = 0 \quad \text{in } (Y^*)^h,$$

and

$$g^h - E^h K^* \xi^h = 0 \quad \text{in } G^h .$$

We make the following hypotheses concerning the approximate operators T^h, $(T^*)^h$, and E^h:

(H13)
$$\lim_{h\to 0} \|(T - T^h)y\|_X = 0 \quad \forall\, y \in Y ,$$

(H14)
$$\lim_{h\to 0} \|(T^* - (T^*)^h)v\|_{Y^*} = 0 \quad \forall\, v \in X^* ,$$

and

(H15)
$$\lim_{h\to 0} \|(E - E^h)s\|_G = 0 \quad \forall\, s \in G^* .$$

We also need the following additional hypotheses on the operators appearing in the definition of the abstract problem:

(H16) *$N \in C^3(X;Y)$ and $\mathcal{F} \in C^3(X;\mathbf{R})$;*

(H17) *N'', N''', $\mathcal{F}''$, and $\mathcal{F}'''$ are locally bounded, i.e., they map bounded sets to bounded sets;*

(H18) *for $v \in X$, in addition to* (H9), *i.e., $N'(v) \in \mathcal{L}(X;Z)$ where $Z \hookrightarrow\hookrightarrow Y$, we have that $[N'(v)]^* \in \mathcal{L}(Y^*;\hat{Z})$ where $\hat{Z} \hookrightarrow\hookrightarrow X^*$, that for $\eta \in Y^*$, $[N''(v)]^* \cdot \eta \in \mathcal{L}(Y^*;\hat{Z})$, and that for $w \in X$, $\mathcal{F}''(v)\cdot w \in \mathcal{L}(X;\hat{Z})$; and*

(H19) *K maps G into Z.*

Here, $(\cdot)''$ and $(\cdot)'''$ denote second and third Fréchet derivatives, respectively. Using hypotheses (H13)-(H19) we can prove the following results. Let $\big(u(\lambda), g(\lambda), \xi(\lambda)\big) \in \mathcal{X}$, for $\lambda \in \Lambda$, be a branch of regular solutions of the optimality system. Then, there exists a $\delta > 0$ and an $h_0 > 0$ such that for $h < h_0$, the discrete optimality system has a unique solution $\big(u^h(\lambda), g^h(\lambda), \xi^h(\lambda)\big)$ satisfying

$$\|\big(u(\lambda), g(\lambda), \xi(\lambda)\big) - \big(u^h(\lambda), g^h(\lambda), \xi^h(\lambda)\big)\|_{\mathcal{X}} < \delta .$$

Moreover,

$$\lim_{h\to 0} \|\big(u(\lambda), g(\lambda), \xi(\lambda)\big) - \big(u^h(\lambda), g^h(\lambda), \xi^h(\lambda)\big)\|_{\mathcal{X}} = 0$$

uniformly in $\lambda \in \Lambda$ and there exists a constant C, independent of h and λ, such that

$$\begin{aligned}\lim_{h\to 0} &\|\big(u(\lambda), g(\lambda), \xi(\lambda)\big) - \big(u^h(\lambda), g^h(\lambda), \xi^h(\lambda)\big)\|_{\mathcal{X}} \\ &\le C\lambda\Big\{\|(T^h - T)\big(N(u(\lambda)) + Kg(\lambda)\big)\|_X + \|(E^h - E)K^*\xi(\lambda)\|_G \\ &\qquad + \|\big((T^*)^h - T^*\big)\Big([N'(u(\lambda))]^*\xi - \mathcal{F}'(u(\lambda))\Big)\|_{Y^*}\Big\} .\end{aligned}$$

These results may be applied to the derivation of error estimates. They basically state that the error in the approximate solution of the optimal control problem is essentially the same as that in the approximations T^h, $(T^*)^h$ and E^h to the linear operators T, T^*, and E. The latter errors can usually be deduced using standard methodologies, e.g., for linear elliptic problems.

The above framework and analyses have been applied to some concrete problems, all of which feature constraints on admissible states and controls that take the form of a system of nonlinear partial differential equations. In each application, we use a different control mechanism so that the discussion provided in [13] illustrates the treatment of a variety of such mechanisms. However, one could use any of the control mechanisms discussed in any of the applications in any other application, or in fact, use any combination of such mechanisms. The first application is to *distributed controls for the von Kármán plate equations*. For this application we will use distributed controls, i.e., control is effected through a source term in the governing partial differential equations. Let Ω be a bounded, convex polygonal domain in $\mathbf{R}^2$ and let Γ denote the boundary of Ω. The von Kármán equations for a clamped plate are given by (after suitable rescaling)

$$\Delta^2\psi_1 + \frac{\lambda}{2}[\psi_2, \psi_2] = 0 \quad \text{in } \Omega\,,$$

$$\Delta^2\psi_2 - \lambda\,[\psi_1, \psi_2] = \lambda\, g \quad \text{in } \Omega\,,$$

and

$$\psi_1 = \frac{\partial\psi_1}{\partial n} = \psi_2 = \frac{\partial\psi_2}{\partial n} = 0 \quad \text{on } \Gamma\,.$$

Here, ψ_1 denotes the (sccaled) Airy stress function, ψ_2 the (scaled) deflection of the plate in the direction normal to the plate, λg is an external load normal to the plate which depends on the loading parameter λ, and $\partial(\cdot)/\partial n$ the normal derivative in the direction of the outer normal to Γ. We define the functional

$$\mathcal{J}(\psi, g) = \mathcal{J}(\psi_1, \psi_2, g) = \frac{\lambda}{2}\int_\Omega \Big((\psi_1 - \psi_{10})^2 + (\psi_2 - \psi_{20})^2\Big)\, d\Omega + \frac{\lambda}{2}\int_\Omega g^2\, d\Omega\,.$$

We then consider the following optimal control problem associated with the von Kármán plate equations:

$$\min\{\,\mathcal{J}(\psi, g) \mid \psi \in \mathcal{Y},\ g \in \Theta\,\} \quad \textit{subject to } (\psi, g) \textit{ satisfying the von Kármán equations}\,,$$

where $\mathcal{Y}$ and Θ are suitable sets. In particular, functions in $\mathcal{Y}$ have square integrable second derivatives. For this problem, the abstract framework can be used to show that optimal solutions exist and that the Lagrange multiplier rule may be used, to derive an optimality system from which optimal states and controls may be deduced, and to derive optimal error estimates for conforming finite element approximations of solutions of the optimality system.

The second application is the Neumann boundary controls of a simplifie model for superconductivity. Let Ω be a bounded open domain in $\mathbb{R}^d$, $d = 2$ or 3, and let Γ be its boundary. A simplified Ginzburg-Landau model for superconductivity is given by (after appropriate rescalings)

$$-\Delta\psi_1 + (|\mathbf{A}|^2 - 1)\psi_1 - \nabla\cdot(\mathbf{A}\psi_2) - \mathbf{A}\cdot\nabla\psi_2 + \lambda\left(\psi_1^2 + \psi_2^2\right)\psi_1 = 0 \quad \text{in } \Omega\,,$$

$$-\Delta\psi_2 + (|\mathbf{A}|^2 - 1)\psi_2 + \nabla\cdot(\mathbf{A}\psi_1) + \mathbf{A}\cdot\nabla\psi_1 + \lambda\left(\psi_1^2 + \psi_2^2\right)\psi_2 = 0 \quad \text{in } \Omega\,,$$

$$\mathbf{n}\cdot(\nabla\psi_1 + \mathbf{A}\psi_2) = \lambda g_1 \quad \text{on } \Gamma\,,$$

and

$$\mathbf{n}\cdot(\nabla\psi_2 - \mathbf{A}\psi_1) = \lambda g_2 \quad \text{on } \Gamma\,.$$

Here, ψ_1 and ψ_2 denote the real and imaginary parts, respectively, of the (rescaled) complex-valued order parameter, $\mathbf{A}$ is a given real magnetic potential, g_1 and g_2 are related to the normal component of the current at the boundary, and $\lambda > 0$ is a "current loading" parameter. Given a desired state $\boldsymbol{\psi}_0 = (\psi_{10}, \psi_{20})$, we define for any $\boldsymbol{\psi} = (\psi_1, \psi_2)$ and $\mathbf{g} = (g_1, g_2)$ the functional

$$\mathcal{J}(\boldsymbol{\psi}, \mathbf{g}) = \frac{\lambda}{2}\int_\Omega \left((\psi_1 - \psi_{10})^2 + (\psi_2 - \psi_{20})^2\right) d\Omega + \frac{\lambda}{2}\int_\Gamma (g_1^2 + g_2^2)\, d\Gamma\,.$$

We then consider the following optimal control problem associated with the Ginzburg-Landau equations for superconductivity:

$$\min\{\, \mathcal{J}(\boldsymbol{\psi}, \mathbf{g}) \mid \boldsymbol{\psi} \in \mathcal{W},\ \mathbf{g} \in \Theta \,\}$$

subject to $(\boldsymbol{\psi}, \mathbf{g})$ *satisfying the Ginburg-Landau equations*,

where $\mathcal{W}$ and Θ are suitable chosen sets; in particular, functions belonging to $\mathcal{W}$ have one square integrable derivative. For this problem, the abstract framework can be used to show that optimal solutions exist and that the Lagrange multiplier rule may be used, to derive an optimality system from which optimal states and controls may be deduced, and to derive optimal error estimates for conforming finite element approximations of solutions of the optimality system.

The third aplication is to the Dirichlet boundary control of the Navier-Stokes equations of incompressible, viscous flow. Let Ω denote a bounded domain in $\mathbb{R}^d$, $d = 2$ or 3, with a boundary denoted by Γ. Let $\mathbf{u}$ and p denote the velocity and pressure fields in Ω. The Navier-Stokes equations for a viscous, incompressible flow are given by (after appropriate rescalings)

$$-\nabla\cdot\left((\nabla\mathbf{u}) + (\nabla\mathbf{u})^T\right) + \nabla p + \lambda\mathbf{u}\cdot\nabla\mathbf{u} = \lambda\mathbf{f} \quad \text{in } \Omega\,,$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\,,$$

and

$$\mathbf{u} = \lambda(\mathbf{b} + \mathbf{g}) \quad \text{on } \Gamma\,.$$

where $\mathbf{f}$ is a given body force, $\mathbf{b}$ and $\mathbf{g}$ are boundary velocity data with $\int_\Gamma \mathbf{b}\cdot\mathbf{n}\,d\Gamma = 0$ and $\int_\Gamma \mathbf{g}\cdot\mathbf{n}\,d\Gamma = 0$, and ν denotes the (constant) kinematic viscosity. We have absorbed the constant density into the pressure and the body force. If the variables in these equations are nondimensionalized, then λ is simply the Reynolds number Re. Given a desired velocity field $\mathbf{u}_0$, we define for any $\mathbf{u}$ and $\mathbf{g}$ the functional

$$\mathcal{J}(\mathbf{u},p,\mathbf{g}) = \frac{\lambda}{4}\int_\Omega |\mathbf{u}-\mathbf{u}_0|^4\,d\Omega + \frac{\lambda}{2}\int_\Gamma (|\nabla_s \mathbf{g}|^2 + |\mathbf{g}|^2)\,d\Gamma\,,$$

where ∇_s denotes the surface gradient. We then consider the following optimal control problem associated with the Navier-Stokes equations:

$$\min\{\mathcal{J}(\mathbf{u},p,\mathbf{g}) : (\mathbf{u},p)\in\mathcal{Z},\ \ \mathbf{g}\in\Theta\}$$

subject to $(\mathbf{u},\mathbf{g})$ *satisfying the Navier-Stokes equations*,

where $\mathcal{Z}$ and Θ are suitable sets. For this problem, the abstract framework can be used to show that optimal solutions exist and that the Lagrange multiplier rule may be used, to derive an optimality system from which optimal states and controls may be deduced, and to derive optimal error estimates for conforming finite element approximations of solutions of the optimality system.

Feedback control of Karman vortex shedding

The control of the forces, e.g., lift and drag, exerted on a submerged obstacle by the fluid that surrounds it is important in many applications. Here we focus on the problem of controlling the lift in the plane, unsteady, incompressible, viscous flow around a circular cylinder. The control of flows has a long and rich history. Some of these past efforts have been devoted to conducting experiments and simulations with different controlling mechanisms, without any attempt to optimally design or to actively change these mechanisms. For example, one may consult textbooks for detailed expositions of such efforts in the area of boundary layer control. Of course, there has been great success in the design and application of active controls, especially in the area of aerodynamic controls. These efforts, for the most part, use simple models to account for the effect of the flow on the submerged object.

Flows about a cylinder at even moderate values of Reynolds numbers, e.g., $Re > 47$, exhibit an unsymmetric, periodic shedding of vortices. As a result, the lift force exerted by the fluid on the cylinder is periodic in time. Here, our goal is to show the efficacy of a simple feedback control law for the reduction of the magnitude of the lift. The control mechanism used to attempt to the reduce the size of the lift oscillations is the injection and suction of fluid through orifices on the surface of the cylinder. The amount of fluid injected or sucked through the orifices is determined, using a simple feedback law, from the pressure "measured" at various stations on the cylinder. Thus, in the language of

feedback control, the *sensor* determines the pressure at various stations on the cylinder and the *actuator* injects or sucks fluid through orifices also on the cylinder. Other sensing and actuating mechanisms could also be used, e.g., the vorticity or stress components on the cylinder for sensing and rotation or shape modification of the cylinder for actuating.

The type of feedback laws used in our study are based on the observation that differences in the pressure on the top and bottom halves of the cylinder should give an indication of the asymmetric behavior of the lift. Such pressure differences are then used to determine how much fluid to inject or suck through the orifices. No attempt is made to systematically design an "optimal" feedback law, i.e., one that in some sense does the best possible job in meeting the objective. However, the computational results we have obtained indicate that the simple feedback law we use here is quite effective in reducing the size of the oscillations in the lift. Details may be found in [16] and [18].

We have investigated the control problem by numerically solving the 2-D Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \frac{1}{Re}\nabla^2\mathbf{u} \qquad \text{and} \qquad \nabla\cdot\mathbf{u} = 0\,,$$

where $\mathbf{u}$ and p are the velocity and kinematic pressure (pressure divided by density), respectively. The above equations are nondimensionalized using the cylinder diameter d, the free stream velocity U, and the kinematic viscosity of the fluid ν. The Reynolds number Re is defined by $Re = Ud/\nu$. Initial and boundary conditions on the cylinder surface are imposed on the velocity:

$$\mathbf{u}(\mathbf{x},0) = 0 \quad \text{in the computational domain } \Omega$$

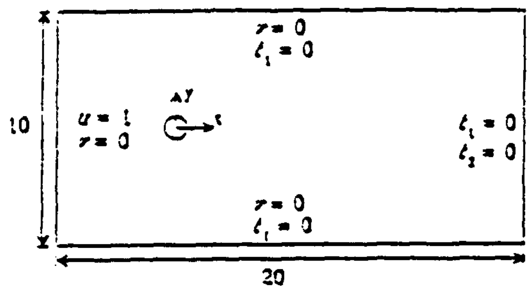
and

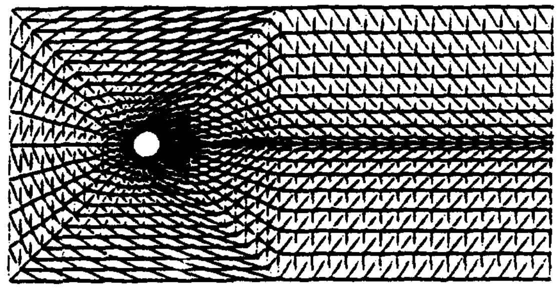
$$\mathbf{u}(\mathbf{x},t) = \mathbf{g} \quad \text{on the cylinder surface } \Gamma \text{ and for } \; t > 0\,.$$

For the uncontrolled flow about the cylinder, the cylinder surface is a solid wall so that in this case $\mathbf{g} = 0$. When controls are applied, this boundary condition becomes inhomogeneous on the portions of Γ covered by the injection and suction orifices, i.e., $\mathbf{g} \neq 0$ at these orifices.

In order to avoid complications with boundary conditions at infinity, a specific finite domain was defined. The origin of the (x,y) coordinate system is located at the center of the cylinder and the cylinder has a unit diameter. The computational domain we use is the rectangle $-5 \leq x \leq 15$ and $-5 \leq y \leq 5$ with the cylinder excluded. The geometry of the domain is sketched in figure below. The exterior boundary of the computational domain consists of the four sides of the rectangle; again, see the left-hand figure below. At the inflow boundary Γ_i the velocity is set to the uniform value at infinity, i.e., $u = 1$ and $v = 0$, where u and v denote the x and y components of the velocity vector, respectively. At the outflow Γ_o a vanishing "stress" boundary condition is imposed. Specifically, if $\mathbf{t} = -p\mathbf{n} + (1/Re)\partial\mathbf{u}/\partial n$, then on Γ_o we set $t_1 = t_2 = 0$, where t_1 and t_2 respectively denote the x and y components of $\mathbf{t}$. The vector $\mathbf{t}$ is not actually the stress vector; however, it has been found that imposing such an outflow condition is often more effective

than requiring the true stress to vanish. Moreover, it is a more convenient boundary conditions to use with the particular form of the viscous term appearing in the Navier-Stokes equations given above. On the top Γ_t and bottom Γ_b of the rectangle we impose the mixed conditions $v = 0$ and $t_1 = 0$. The boundary conditions are also indicated on the sketch of the computational domain given in left-hand figure below.





For the uncontrolled case, the problem we wish to discretize is then defined by the above equations with boundary conditions on Γ_i, Γ_o, Γ_b, and Γ_t.

The spatial discretization is effected using the Taylor-Hood finite element pair based on a triangulation of the computational flow domain. A typical triangulation is depicted in right-hand figure above; it consists of 1735 triangles and 3735 velocity nodes. The velocity (pressure) field is approximated using continuous piecewise quadratic (linear) polynomials; the same grid is used for both fields.

The semi-implicit single-step Euler method is employed for the time discretization. This time-stepping algorithm is defined as follows: given $\mathbf{u}^{m-1}$, the pair $(\mathbf{u}^m, p^m)$ is obtained by solving the linear system

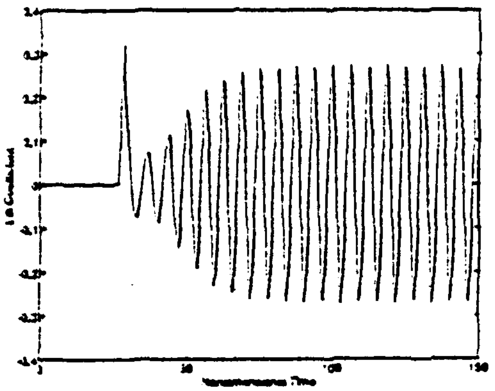
$$\frac{\mathbf{u}^m - \mathbf{u}^{m-1}}{\delta} + (\mathbf{u}^{m-1} \cdot \nabla)\mathbf{u} \qquad \text{and} \qquad \nabla \cdot \mathbf{u}^m = 0\,,$$

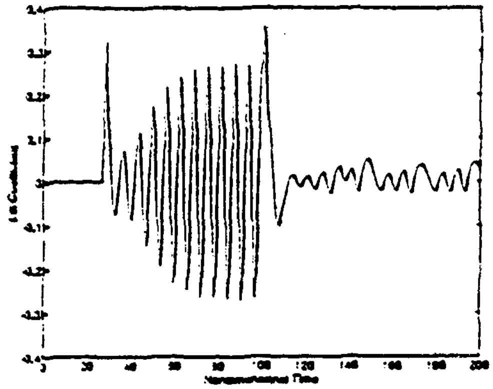
where δ denotes the time step. Of course, (5) and (6) are supplemented by the appropriate boundary and initial conditions. This scheme has been previously analyzed and is found to be unconditionally stabe.

Using the computational scheme described above, we have simulated the uncontrolled problem at $Re = 60$ and $Re = 80$. To break the symmetry in the problem and trigger the vortex shedding, a perturbation is applied for a short time interval. The transient solution develops until a periodic state is reached at approximately $t = 70$ and $t = 90$ for $Re = 60$ and $Re = 80$, respectively. These phases of the solution are perfectly captured by the lift coefficient C_L defined by

$$C_L = \int_0^{2\pi} \tilde{t}_2(\theta)\, d\theta,$$

where $\tilde{t}_2$ is the y component of the true stress vector and θ is the angle along the surface of the cylinder measured from the leading edge. The evolution in time of C_L exhibits an oscillation having a period $T \approx 7.0$ units in time for $Re = 60$ and $T \approx 6.5$ for $Re = 80$, leading to a Strouhal number $St \approx 0.14$ for $Re = 60$ and $St \approx .15$ for $Re = 80$; see the left-hand figure below for $Re = 80$. These values are in agreement with experimental results and other numerical simulations.





We have used a variety of number and location of blowing and suction orifices along the back side of the cylinder. One effective arrangement is to have two suction slots centered at $\pm 23\pi/32$ on the back-side of the cylinder and a single blowing slot centered at π. We place the sensors at symmetric locations on the front-side of the cylinder and we sense the pressure at these locations. The feedback law is chosen as follows:

$$\mathrm{u} = \min\left\{\alpha|p(\theta_1) - p(\theta_2)|, \beta\right\} g(\theta) .$$

Here, α and β are constants, with β being used to limit the size of allowable controls; g is a given velocity profile at the orifices. We have performed computational simulations for numerous cases but report only the case of $\alpha = 30$ and $\beta = 2.0$. The control is turned on at $t = 100$. The results are shown in right-hand figure above from which one can conclude that the simple feedback law can be quite effective in reducing the size of the oscillations in the lift.

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